Rutgers University: Real Variables and Elementary Point-Set Topology Qualifying Exam

August 2017: Problem 3 Part 1 Solution

Exercise. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a non-negative measurable function. Define a function $f : \mathbb{R} \to \mathbb{C}$ by letting

$$f(x) = \int_{-\infty}^{\infty} \frac{e^{ixy}}{1 + y^2 + \sigma(y)} dy \tag{1}$$

Prove that

1. The function f is well defined (that is the integral of (1) exists) and continuous for all $x \in \mathbb{R}$.

Solution.

$$\begin{aligned} f \text{ is well defined } &\iff \frac{e^{ixy}}{1 + y^2 + \sigma(y)} \text{ is integrable} \\ &\iff \int_{-\infty}^{\infty} \frac{e^{ixy}}{1 + y^2 + \sigma(y)} dy < \infty \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty} \left| \frac{e^{ixy}}{1 + y^2 + \sigma(y)} \right| dy &= \int_{-\infty}^{\infty} \frac{1}{1 + y^2 + \sigma(y)} dy && \text{since } |e^{ixy}| = 1 \text{ and } y^2 \geq 0,\ \sigma(y) \geq 0 \\ &\leq \int_{-\infty}^{\infty} \frac{1}{1 + y^2} dy && \text{since } \sigma(y) \geq 0 \\ &= \pi \\ &< \infty \end{aligned}$$

Thus, f is well-defined.

Dominated Comvergence Theorem: Let $\{f_n\}$ be a sequence in L^1 s.t.

i) $f_n \to f$ a.e. and

ii) $\exists$a non-negative $g \in L^1$ s.t. $|f_n| \leq g$ a.e. $\forall n$

Then $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$

i) Let $h(y) = \frac{e^{ixy}}{1+y^2+\sigma(y)}$ and $h_n(y) = \frac{e^{ix_n y}}{1+y^2+\sigma(y)}$ s.t. $(x_n) \subseteq \mathbb{R}$ and $(x_n) \to x_0$. Then $\lim_{n\to\infty} h_n(y) = h(y)$ since e^{ixy} is continuous relative to f.

ii) Also, $|h_n(y)| = \left| \frac{e^{ix_n y}}{1+y^2+\sigma(y)} \right| = \frac{1}{1+y^2+\sigma(y)} \leq \frac{1}{1+y^2}$ for all n
And, as shown above, $\frac{1}{1+y^2}$ us a non-negative integrable function.

Thus, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int h_n(y) dy = \int h(y) dy$$

(Solution continued on next page)

Solution.

$$
\begin{aligned}
\implies \lim_{x\to x_0} f(x) &= \lim_{x\to x_0} \int_{-\infty}^{\infty} \left| \frac{e^{ixy}}{1+y^2+\sigma(y)} \right| dy \\
&= \lim_{n\to\infty} \int_{-\infty}^{\infty} \left| \frac{e^{ix_n y}}{1+y^2+\sigma(y)} \right| dy \\
&= \int_{-\infty}^{\infty} \left| \frac{e^{ix_0 y}}{1+y^2+\sigma(y)} \right| dy \\
&= f(x_0)
\end{aligned}
$$

So f is continuous at x_0, but $x_0 \in \mathbb{R}$ was arbitrary.
Thus, f is continuous for all $x \in \mathbb{R}$.